Tuesday, April 04, 2017
8:11 AM

Consider first:

$R_{tr} = \frac{2}{r}\frac{\partial B}{\partial t} = 0 \quad \Rightarrow B(r,t) = B(r)$

then

$R_{\theta\theta} = e^{-2B}\left[r\left(\frac{\partial B}{\partial r} - \frac{\partial \alpha}{\partial r}\right) - 1\right] + 1 = 0$

$\Downarrow$

$\partial_t R_{\theta\theta} = -2\cancel{\frac{\partial B}{\partial t}}^{0} e^{-2B}[\quad] + e^{-2B}\left[r\cancel{\frac{\partial^2 B}{\partial t\partial r}}^{0} - r\frac{\partial^2 \alpha}{\partial t \partial r}\right] = 0$

hence

$-re^{-2B}\partial_t\partial_r \alpha(r,t) = 0 \quad \Rightarrow \alpha(r,t) = f(r) + g(t)$

Then:

$ds^2 = -e^{2f(r)}e^{2g(t)}dt^2 + e^{2B(r)}dr^2 + r^2 d\Omega^2$

But now we redefine $t \to t' = \int e^{g(t)}dt \Rightarrow dt' = e^{g(t)}dt$ to get:

$ds^2 = -e^{2f(r)}dt'^2 + e^{2B(r)}dr^2 + r^2 d\Omega^2$ Hey guys! I'm diagonal :)

Notice anything unusual? None of the metric components depend on t'!
That is, without assuming time-independence, we were able to show that the metric (in an appropriate set of coordinates) does not depend on time, i.e. $g_{\mu\nu}(r,t) = g_{\mu\nu}(r)$.

Such a space is called stationary (not changing). It also implies the existence of a timelike Killing vector ∂_t.

But this space is even gooder, because it is also invariant under $t \to -t$ due to the absence of any $dx^i dt$ cross-terms. This extra condition makes the space static (not doing anything).

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To understand the difference, consider a planet which is sitting still vs. one that is spinning. The former would create a static geometry while the latter would only create a stationary geometry, since $t \to -t$ reverses the spin.

Thus our assumption of a spherically symmetric source free solution implies a static geometry.

To finish up we need the actual form of $f(r)$ and $\beta(r)$.
Returning to $R_{\mu\nu} = 0$ w/ everything we know so far:

$$R_{tt} = e^{2(f-\beta)}\left[\frac{\partial^2 f}{\partial r^2} + \left(\frac{\partial f}{\partial r}\right)^2 - \frac{\partial f}{\partial r}\frac{\partial \beta}{\partial r} + \frac{2}{r}\frac{\partial f}{\partial r}\right] = 0 \quad \Rightarrow \quad [\ ] = 0$$
$$R_{rr} = -\frac{\partial^2 f}{\partial r^2} - \left(\frac{\partial f}{\partial r}\right)^2 + \frac{\partial f}{\partial r}\frac{\partial \beta}{\partial r} + \frac{2}{r}\frac{\partial \beta}{\partial r} = 0$$

But: $$\underset{0}{[\ ]} + \underset{0}{R_{rr}} = \frac{2}{r}\left(\frac{\partial f}{\partial r} + \frac{\partial \beta}{\partial r}\right) = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial r} = -\frac{\partial \beta}{\partial r}$$
$$\Downarrow$$
$$f(r) = -\beta(r) + c$$

Then:
$$ds^2 = -e^{-2\beta(r)}\overbrace{e^{2c}dt^2}^{=dt^2 \text{ after redefining } t} + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$$

Finally:
$$R_{\theta\theta} = e^{2f}\left(-2r\frac{\partial f}{\partial r} - 1\right) + 1 = 0$$
or $$\frac{\partial}{\partial r}\left(re^{2f}\right) = 1 \quad \Rightarrow \quad re^{2f} = r + c$$
$$e^{2f} = 1 + \frac{c}{r} \quad \Rightarrow \quad e^{2\beta} = \left(1 + \frac{c}{r}\right)^{-1}$$

So:
$$ds^2 = -\left(1 + \frac{c}{r}\right)dt^2 + \left(1 + \frac{c}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

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But we know from the Newtonian limit that $g_{00} = -(1+2\phi) = -(1-\frac{2GM}{r})$

$\phi = -\frac{GM}{r}$

so finally:

$$\boxed{ds^2 = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2}$$

This is the Schwarzschild geometry expressed in Schwarzschild coordinates. This is the geometry <u>outside</u> of any spherically symmetric mass, e.g. planet, star, black hole.

We call $R = 2GM$ the Schwarzschild radius.

For $r \to \infty$ or $M \to 0$ $ds^2 \to -dt^2 + dr^2 + r^2 d\Omega^2 \simeq \mathbb{M}^4$ as expected!

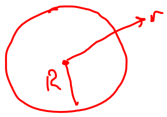
Tuesday, March 24, 2015
6:57 PM

Interior solutions for extended objects

Back to E+M:

We solved $\vec{\nabla}\cdot\vec{E}=0$ for $\vec{E}_{out}(r)=\frac{Q_{tot}}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}$

We now want to solve $\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$ for $\vec{E}_{in}(r)$ again assuming spherical symmetry.

Consider 

and assume $\rho(r)=\underbrace{\begin{cases}\rho_* & r\le R\\ 0 & r>R\end{cases} \Rightarrow Q_{tot}=\frac{4}{3}\pi R^3\rho_*}_{}$

This is a model of the charge distribution. We could have started w/ a different one and end up w/ a different $\vec{E}_{in}(r)$.

To solve for $\vec{E}_{in}(r)$ we integrate over a spherical volume centered at the origin w/ $r<R$:

$$\int\vec{\nabla}\cdot\vec{E}\,d^3x=\int\frac{\rho_*}{\varepsilon_0}d^3x$$

$$\oint\vec{E}_{in}\cdot d\vec{a}=\frac{4}{3}\pi r^3\frac{\rho_*}{\varepsilon_0}=\frac{Q_{tot}}{\varepsilon_0}\frac{r^3}{R^3}$$

$$4\pi r^2E=\frac{Q_{tot}}{\varepsilon_0}\frac{r^3}{R^3}$$

$$\vec{E}_{in}(r)=\frac{Q_{tot}}{4\pi\varepsilon_0}\frac{r}{R^3}\hat{r}$$

Note:

a) The functional dependence on r is different for $\vec{E}_{in}\sim r$ and $\vec{E}_{out}\sim\frac{1}{r^2}$.

b) The solutions agree at the boundary, i.e. $\vec{E}_{in}(r=R)=\vec{E}_{out}(r=R)$.

Thursday, April 06, 2017
7:43 AM

Back to GR:

So our job now is to solve Einstein's equations in a region where $T_{\mu\nu} \neq 0$ and find a solution $g_{\mu\nu}(x)$ that matches the Schwarzchild solution at the boundary.

First issue is that we have to go back to $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$ which is considerably more complicated than $R_{\mu\nu} = 0$. Trace reversing doesn't help either.

However using spherical symmetry we can immediately adopt many of the results from our exterior analysis.

$$ds^2 = -e^{2\alpha(r,t)} dt^2 + e^{2\beta(r,t)} dr^2 + r^2 d\Omega^2 \quad \text{(We got here w/out ever using } R_{\mu\nu} = 0\text{)}$$

Now for the exterior case we found that we could eliminate time-dependence after applying $R_{\mu\nu} = 0$. For the interior this will not generically be the case, but we will <u>assume</u> a model of the interior that does not change w/ time and hence <u>may</u> also assume that our solution will be time independent. This must be checked for consistency at the end of the calculation.

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2 \Rightarrow g_{\mu\nu} = \begin{pmatrix} -e^{2\alpha} & & & \\ & e^{2\beta} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} -e^{-2\alpha} & & & \\ & e^{-2\beta} & & \\ & & r^{-2} & \\ & & & r^{-2}\sin^{-2}\theta \end{pmatrix}$$

We can feed this metric into Mathematica and have it compute $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. The output will include nontrivial expressions for G_{tt}, G_{rr}, $G_{\theta\theta}$, $G_{\phi\phi}$ and the rest of the components will be identically 0. So $G_{\mu\nu}$ will be diagonal, however don't take this too far... a diagonal $g_{\mu\nu}$ does not always yield a diagonal $G_{\mu\nu}$!

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The right hand side of EE is where we put in information about the source, i.e. $T_{\mu\nu}$.

Assume a perfect fluid source : $T_{\mu\nu} = (\rho + p)\, U_\mu U_\nu + p\, g_{\mu\nu}$
To completely specify the source we would also need an equation of state relating ρ and p.
We will keep this general for now.

Recall that U_μ is the overall fluid dual 4-velocity.
We know $\underbrace{U_\mu U^\mu = -1}_{\text{always!}} = g^{\mu\nu} U_\mu U_\nu = \underbrace{-e^{-2\alpha} U_0 U_0}_{\substack{\text{assume we work in coordinates} \\ \text{where the overall 3-velocity of} \\ \text{the source is } \vec{0}\text{, i.e. } U_i = 0.}} \Rightarrow U_0 = e^{\alpha}$

Then: $T_{\mu\nu} = \begin{pmatrix} e^{2\alpha}\rho & & & \\ & e^{2\beta}p & & \\ & & r^2 p & \\ & & & r^2 \sin^2\theta\, p \end{pmatrix}$ where $\alpha(r), \beta(r), \rho(r), p(r)$ are unknown.

Motivated by the Schwarzschild solution we will define:

$e^{2\beta(r)} = \left[1 - \frac{2GM(r)}{r}\right]^{-1}$ which just exchanges the unknown $\beta(r)$ for $M(r)$.

then
$M(r) = \frac{r}{2G}\left[1 - e^{-2\beta(r)}\right]$

To see the value of this consider $G_{tt} = 8\pi G\, T_{tt}$:
$r^{-2}\left[e^{2\alpha - 2\beta}(-1 + e^{2\beta} + 2r\beta')\right] = 8\pi G e^{2\alpha}\rho$

$\beta \to M$ $\quad \frac{dM}{dr} = 4\pi r^2 \rho$

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In the latter form we can easily integrate:

① $m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'$

Since $m(r)$ plays the same role that M plays in Schwarzchild, we can immediately impose a matching condition:

$m(R) = M = 4\pi \int_0^R \rho(r') r'^2 dr'$

Now considering $G_{rr} = 8\pi G T_{rr}$ we get:

$r^{-2}\left[1 - e^{2\beta} + 2r\alpha'\right] = 8\pi G e^{2\beta} p$

$\beta \to m$

② $\alpha' = \dfrac{G m(r) + 4\pi G r^3 p}{r[r - 2G m(r)]}$

Instead of considering the rest of EE we can instead appeal to energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$ which implies:

$(\rho + p)\dfrac{d\alpha}{dr} = -\dfrac{dp}{dr}$ (from the $\nu = r$ term)

Combining this w/ the rr equation we can write:

$\dfrac{dp}{dr} = -\dfrac{(\rho + p)\left[G m(r) + 4\pi G r^3 p\right]}{r[r - 2G m(r)]}$ or the $\underbrace{\text{Tolman-Oppenheimer-Volkoff}}_{\text{T.O.V.}}$ eqn.

The importance of the T.O.V. eqn. is that it supplies us w/ an equation of state.

If we start w/ $\rho(r)$ we can:

a) Use 1 to find $m(r)$ (hence $\beta(r)$)

b) Use 3 to find $p(r)$

c) Use 2 to find $\alpha(r)$

Then we're done!

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As an example consider a constant density star (analogous to our E&M example).

$$\rho(r) = \begin{cases} \rho_* & r \le R \\ 0 & r > R \end{cases}$$

a) $$M(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_* & r \le R \\ \frac{4}{3}\pi R^3 \rho_* = M & r > R \end{cases} \Rightarrow e^{2\beta(r)} = \left[1 - \frac{8\pi G \rho_* r^2}{3}\right]^{-1}$$

b) $$\frac{dp}{dr} = -\frac{(\rho_* + p)\left[G\frac{4}{3}\pi r^3 \rho_* + 4\pi G r^3 p\right]}{r\left[r - 2G\frac{4}{3}\pi r^3 \rho_*\right]}$$

$\Downarrow$ after a "little" work

$$p(r) = \rho_* \left\{ \frac{R\sqrt{R - 2GM} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3R\sqrt{R - 2GM}} \right\} \qquad r < R$$

c) Solving for $\alpha(r)$ we get: $$e^{\alpha(r)} = \frac{3}{2}\left(1 - \frac{2GM}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2GMr^2}{R^3}\right)^{1/2} \qquad r < R$$

Note: @ $r = R$ $e^{\alpha(R)} = \left(1 - \frac{2GM}{R}\right)^{1/2} \Rightarrow e^{2\alpha(R)} = \left(1 - \frac{2GM}{R}\right)$ agreeing w/ Schwarzchild!
We already know that $e^{2\beta(r)}$ matches.

Consider the $p(r)$ expression:

- $p(r)$ increases as r decreases (makes sense!)
- The pressure at $r = 0$ blows up as $M \to \frac{4}{9}\frac{R}{G}$ (from below), that is $R^{3/2} - 3R\sqrt{R - 2GM} \to 0$.
- Therefore if $M > \frac{4}{9}\frac{R}{G}$ then this solution is inconsistent. The consistency failure comes from our static assumption. If $M > \frac{4}{9}\frac{R}{G}$ the system will evolve w/ time!